Explicit formulas and global uniqueness for phaseless inverse scattering in multidimensions

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Abstract. We consider phaseless inverse scattering for the Schrödinger equation with compactly supported potential in dimension $d \geq 2$. We give explicit formulas for solving this problem from appropriate data at high energies. As a corollary, we give also a global uniqueness result for this problem with appropriate data on a fixed energy neighborhood.

1. Introduction

We consider the Schrödinger equation

$$-\Delta\psi + v(x)\psi = E\psi, \quad x \in \mathbb{R}^d, \ d \geq 2, \ E > 0, \tag{1.1}$$

where

$$\begin{aligned} &v \in L^\infty(\mathbb{R}^d), \quad supp\, v \subset D, \\ &D \ \text{ is an open bounded domain in } \ \mathbb{R}^d. \end{aligned} \tag{1.2}$$

For equation (1.1) we consider the classical scattering solutions $\psi^+ = \psi^+(x,k)$, $x \in \mathbb{R}^d$, $k \in \mathbb{R}^d$, $k^2 = E$, specified by the following asymptotics as $|x| \to \infty$:

$$\begin{aligned} &\psi^+(x,k) = e^{ikx} + c(d,|k|)\frac{e^{i|k||x|}}{|x|^{(d-1)/2}} f(k,|k|\frac{x}{|x|}) + O\big(\frac{1}{|x|^{(d+1)/2}}\big), \\ &c(d,|k|) = -\pi i(-2\pi i)^{(d-1)/2}|k|^{(d-3)/2}, \end{aligned} \tag{1.3}$$

for some a priori unknown f. In addition, the function $f = f(k,l)$, $k, l \in \mathbb{R}^d$, $k^2 = l^2 = E$, arising in (1.3) is the classical scattering amplitude for equation (1.1).

In order to find ψ^+ and f from v one can use, in particular, the Lippmann-Schwinger integral equation

$$\begin{aligned} &\psi^+(x,k) = e^{ikx} + \int_D G^+(x-y,k)v(y)\psi^+(y,k)dy, \\ &G^+(x,k) = -(2\pi)^{-d}\int_{\mathbb{R}^d} \frac{e^{i\xi x}d\xi}{\xi^2 - k^2 - i0}, \end{aligned} \tag{1.4}$$

and the formula

$$f(k,l) = (2\pi)^{-d}\int_D e^{-ily}v(y)\psi^+(y,k)dy, \tag{1.5}$$

where $x, k, l \in \mathbb{R}^d$, $k^2 = l^2 = E > 0$; see e.g. [BS], [F2].

The scattering amplitude f for equation (1.1) at fixed E is defined on

$$\mathcal{M}_E = \{k \in \mathbb{R}^d,\ l \in \mathbb{R}^d : \quad k^2 = l^2 = E\}, \quad E > 0. \tag{1.6}$$

In addition to f on $\mathcal{M}_E$, we consider also $f\big|_{\Gamma_E}$, where

$$\begin{aligned}&\Gamma_E = \{k = k_E(p),\ l = l_E(p) :\ p \in \bar{\mathcal{B}}_{2\sqrt{E}}\},\\ &k_E(p) = \frac{p}{2} + \big(E - \frac{p^2}{4}\big)^{1/2}\gamma(p),\ l_E(p) = -\frac{p}{2} + \big(E - \frac{p^2}{4}\big)^{1/2}\gamma(p),\end{aligned} \tag{1.7}$$

$$\mathcal{B}_r = \{p \in \mathbb{R}^d :\ |p| < r\}, \quad \bar{\mathcal{B}}_r = \{p \in \mathbb{R}^d :\ |p| \le r\}, \quad r > 0, \tag{1.8}$$

where γ is a piecewise continuous vector-function on $\mathbb{R}^d$ such that

$$|\gamma(p)| = 1, \quad \gamma(p)p = 0, \quad p \in \mathbb{R}^d. \tag{1.9}$$

One can see that

$$\Gamma_E \subset \mathcal{M}_E,\ dim\,\Gamma_E = d, \quad dim\,\mathcal{M}_E = 2d-2,\ E > 0,\ d \ge 2. \tag{1.10}$$

Let

$$\mathcal{M}_\Lambda = \cup_{E\in\Lambda}\mathcal{M}_E, \quad \Gamma_\Lambda = \cup_{E\in\Lambda}\Gamma_E, \tag{1.11}$$

where $\Lambda \subseteq \mathbb{R}_+ =]0, +\infty[$.

We start with the following inverse scattering problems for equation (1.1) under assumptions (1.2):

Problem 1.1. Reconstruct potential v on $\mathbb{R}^d$ from its scattering amplitude f on some appropriate $\mathcal{M}' \subseteq \mathcal{M}_{\mathbb{R}_+}$.

Problem 1.2. Reconstruct potential v on $\mathbb{R}^d$ from its phaseless scattering data $|f|^2$ on some appropriate $\mathcal{M}' \subseteq \mathcal{M}_{\mathbb{R}_+}$.

Note that in quantum mechanical scattering experiments (in framework of model described by equation (1.1)) the phaseless scattering data $|f|^2$ can be measured directly, whereas the complete scattering amplitude f is not accessible for direct measurements. Therefore, Problem 1.2 is of particular interest from applied point of view in the framework of quantum mechanical inverse scattering. However, in the literature much more results are given on Problem 1.1 (see [ABR],[B], [BAR], [ChS], [EW], [E], [ER], [F1], [F2], [G], [HH], [HN], [I], [IN], [Me], [Mo], [Ne], [N1]-[N7], [R], [S], [VW], [WY] and references therein) than on Problem 1.2 (see chapter X of [ChS] and recent works [K1], [K2] and references therein, where in [K1], [K2] some similar problem is considered).

In particular, for Problem 1.1 it is well known that the scattering amplitude f at high energies uniquely determines v via the formulas

$$\hat{v}(k-l) = f(k,l) + O(E^{-1/2}) \ \text{ as } \ E \to +\infty, \quad (k,l) \in \mathcal{M}_E, \tag{1.12}$$

$$\hat{v}(p) = (2\pi)^{-d}\int\limits_D e^{ipx}v(x)dx, \quad p \in \mathbb{R}^d; \tag{1.13}$$

see, for example, [F1], [N7].

On the other hand, for Problem 1.2 it is well known that the phaseless scattering data $|f|^2$ on $\mathcal{M}_{\mathbb{R}_+}$ do not determine v uniquely, in general. In particular, we have that

$$\begin{aligned} &f_y(k,l) = e^{i(k-l)y} f(k,l), \\ &|f_y(k,l)|^2 = |f(k,l)|^2, \quad (k,l) \in \mathcal{M}_{\mathbb{R}_+}, \quad y \in \mathbb{R}^d, \end{aligned} \tag{1.14}$$

where f is the scattering amplitude for v and f_y is the scattering amplitude for v_y, where

$$v_y(x) = v(x-y), \quad x \in \mathbb{R}^d, \quad y \in \mathbb{R}^d; \tag{1.15}$$

see, for example, Lemma 1 of [N6].

In the present work, in view of the aforementioned nonuniqueness for Problem 1.2 we modify this problem into Problem 1.3 formulated below. Let

$$S = \{|f|^2, |f_j|^2, \quad j = 1, \dots, n\}, \tag{1.16}$$

where f is the initial scattering amplitude for v satisfying (1.2) and f_j is the scattering amplitude for

$$v_j = v + w_j, \quad j = 1, \dots, n, \tag{1.17}$$

where $w_1, \dots, w_n$ are additional a priori known background scatterers such that

$$\begin{aligned} &w_j \in L^\infty(\mathbb{R}^d), \quad supp\, w_j \subset \Omega_j, \\ &\Omega_j \ \text{ is an open bounded domain in } \ \mathbb{R}^d, \ \Omega_j \cap D = \emptyset, \end{aligned} \tag{1.18a}$$

$$\begin{aligned} &w_j \neq 0, \ w_{j_1} \neq w_{j_2} \ \text{ for } \ j_1 \neq j_2 \ \text{ in } \ L^\infty(\mathbb{R}^d), \\ &j, \ j_1, \ j_2 \in \{1, \dots, n\}. \end{aligned} \tag{1.18b}$$

Problem 1.3. Reconstruct potential v on $\mathbb{R}^d$ from the phaseless scattering data S on some appropriate $\mathcal{M}' \subseteq \mathcal{M}_{\mathbb{R}_+}$ and for some appropriate background scatterers $w_1, \dots, w_n$.

Note also that Problems 1.1, 1.2, 1.3 can be considered as examples of ill-posed problems; see [LRS] for an introduction to this theory.

Problem 1.3 in dimension $d = 1$ was, actually, considered in [AS] for $n = 1$. However, to our knowledge, Problem 1.3 in dimension $d \geq 2$ was not yet considered in the literature before the present work.

Results of the present work can be summarized as follows.

First, we give explicit formulas for solving Problem 1.3 in dimension $d \geq 2$ for $n = 2$ and $\mathcal{M}' = \Gamma_\Lambda$ defined by (1.7), (1.11) for any unbounded $\Lambda \subset \mathbb{R}_+$; see Theorem 2.1, Remark 3.1 and Corollary 2.1 of Section 2. As an example of Λ for this result one can take $\Lambda = [E_0, +\infty[, E_0 > 0$, or just Λ of Remark 2.1.

Second, we give a global uniqueness result for Problem 1.3 in dimension $d \geq 2$ for $n = 2$ and $\mathcal{M}' = \Gamma_\Lambda$ for any bounded infinite $\Lambda \subset \mathbb{R}_+$; see Theorem 2.2 of Section 2. As an example of Λ for this result one can take $\Lambda =]E_0 - \varepsilon, E_0 + \varepsilon[, E_0 > 0,\ \varepsilon > 0,\ E_0 - \varepsilon \geq 0$, or just Λ of Theorem 2.2.

In addition, we indicate possible extensions of the aforementioned results to the case $n = 1$; see Propositions 2.1, 2.2 of Section 2.

The progress of the present work in comparison with the recent works [K1], [K2] includes explicit formulas for phaseless inverse scattering at high energies and no assumption that $v \geq 0$. In addition, in the present work we consider inverse scattering from far field phaseless scattering data (and not from near field phaseless scattering data as in [K1], [K2]).

The main statements of the present work are presented in detail in the next section.

2. Main statements

2.1. Notations and related remarks. Let

$$\hat{u}(p) = (2\pi)^{-d} \int\limits_{\mathbb{R}^d} e^{ipx} u(x) dx, \quad p \in \mathbb{R}^d, \tag{2.1}$$

where u is a test function on $\mathbb{R}^d$. In particular, we consider $\hat{u} = \hat{v},\ \hat{w}_j$ for $u = v,\ w_j$, $j = 1, \ldots, n$, where v, w_j satisfy (1.2), (1.18).

Note that if

$$u_y(x) = u(x - y), \quad x, y \in \mathbb{R}^d, \tag{2.2a}$$

then

$$\hat{u}_y(p) = e^{ipy} \hat{u}(p), \quad p \in \mathbb{R}^d. \tag{2.2b}$$

We represent $\hat{v}$ and $\hat{w}_j$ as follows:

$$\begin{aligned} &\hat{v}(p) = |\hat{v}(p)|\theta(p), \quad \theta(p) = e^{i\alpha(p)}, \\ &\hat{w}_j(p) = |\hat{w}_j(p)|\omega_j(p), \quad \omega_j(p) = e^{i\beta_j(p)}, \end{aligned} \tag{2.3}$$

where $p \in \mathbb{R}^d$, $j = 1, \ldots, n$.

We consider the following sets:

$$A_y = \{p \in \mathbb{R}^d : \quad e^{2ipy} = 1\}, \quad y \in \mathbb{R}^d, \tag{2.4}$$

$$Z_0 = \{p \in \mathbb{R}^d : \quad |\hat{v}(p)| = 0\},\ Z_j = \{p \in \mathbb{R}^d : \quad |\hat{w}_j(p)| = 0\}, \quad j = 1, \ldots, n, \tag{2.5}$$

$$Y_{j_1,j_2} = \{p \in \mathbb{R}^d \backslash (Z_{j_1} \cup Z_{j_2}) : (\omega_{j_1}(p))^2 = (\omega_{j_2}(p))^2\}, \quad 1 \leq j_1, j_2 \leq n,\ j_1 \neq j_2. \tag{2.6}$$

We have, in particular, that

$$A_y \ \text{ is closed and } \ Mes\, A_y = 0 \ \text{ in } \ \mathbb{R}^d,\ y \neq 0. \tag{2.7}$$

Assumptions (1.2) on v imply, in particular, that $\hat{v}$ is (complex-valued) real-analytic on $\mathbb{R}^d$. Therefore:

$$Z_0 \ \text{ is closed in } \ \mathbb{R}^d; \ \ Mes\, Z_0 = 0 \ \text{ in } \ \mathbb{R}^d \ \text{ if } \ \hat{v} \not\equiv 0. \tag{2.8}$$

Assumptions (1.18) on w_j imply, in particular, that $\hat{w}_j$ is (complex-valued) real-analytic on $\mathbb{R}^d$ and $\hat{w}_j \not\equiv 0$, $j = 1, \ldots, n$. Therefore,

$$Z_j \ \text{ is closed and } \ Mes\, Z_j = 0 \ \text{ in } \ \mathbb{R}^d, \ j = 1, \ldots, n. \tag{2.9}$$

In addition, if

$$w_j(x) = w_j^0(|x-y|), \quad x \in \mathbb{R}^d, \quad \text{for some } \ w_j^0, \tag{2.10}$$

for some j and some $y \in \mathbb{R}^d$, then

$$Z_j = \{p \in \mathbb{R}^d : \quad |p| \in \mathcal{R}_j\}, \tag{2.11}$$

where $\mathcal{R}_j$ is a discrete set in $\mathbb{R}_+$ without accumulation points (except $+\infty$) and $\mathcal{R}_j$ is independent of y.

In addition, taking into account (2.2), if

$$w_{j_2}(x) = w_{j_1}(x-y), \quad x \in \mathbb{R}^d, \quad j_2 \neq j_1, \tag{2.12}$$

for some j_1, j_2 and some $y \in \mathbb{R}^d \backslash \{0\}$, then

$$Y_{j_1,j_2} \subseteq A_y. \tag{2.13}$$

2.2. Results on Problem 1.3 in dimension $d \geq 2$ for $n = 2$.

Theorem 2.1. *Suppose that complex-valued v satisfies (1.2), complex-valued w_j satisfies (1.18a), $j = 1, 2$, $d \geq 2$. Then the following formulas hold:*

$$|\hat{v}_j(p)|^2 = \lim_{\substack{p=k-l,(k,l)\in\mathcal{M}_E,\\ E\to+\infty}} |f_j(k,l)|^2 \ \text{ for each } \ p \in \mathbb{R}^d, \ j = 0, 1, 2, \tag{2.14}$$

$$\begin{aligned} &||\hat{v}_j(p)|^2 - |f_j(k,l)|^2| \leq c(D_j) N_j^3 E^{-1/2}, \\ &p = k - l, \quad (k,l) \in \mathcal{M}_E, \ E^{1/2} \geq \rho(D_j, N_j), \ j = 0, 1, 2, \end{aligned} \tag{2.15}$$

where $v_0 = v$, $f_0 = f$, $D_0 = D$, v_j is defined by (1.17) and $D_j = D \cup \Omega_j$ for $j = 1, 2$, $\|v_j\|_{L^\infty(D_j)} \leq N_j$, $j = 0, 1, 2$, and c, ρ are defined by (3.10), (3.11). Suppose, in addition, that w_1, w_2 satisfy (1.18b) and that

$$Mes\, \bar{Y}_{1,2} = 0 \ \text{ in } \ \mathbb{R}^d, \tag{2.16}$$

where $\bar{Y}_{1,2}$ denotes the closure of $Y_{1,2}$ in $\mathbb{R}^d$. Then the following formula holds:

$$\begin{aligned} &\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = (\sin(\beta_2 - \beta_1))^{-1} \times \\ &\begin{pmatrix} \sin\beta_2 & -\sin\beta_1 \\ -\cos\beta_2 & \cos\beta_1 \end{pmatrix} \begin{pmatrix} (2|\hat{v}||\hat{w}_1|)^{-1}(|\hat{v}_1|^2 - |\hat{v}|^2 - |\hat{w}_1|^2) \\ (2|\hat{v}||\hat{w}_2|)^{-1}(|\hat{v}_2|^2 - |\hat{v}|^2 - |\hat{w}_2|^2) \end{pmatrix}, \end{aligned} \tag{2.17}$$

$\alpha = \alpha(p)$, $|\hat{v}| = |\hat{v}(p)|$, $\beta_j = \beta_j(p)$, $|\hat{w}_j| = |\hat{w}_j(p)|$, $j = 1, 2$, $p \in \mathbb{R}^d \backslash (Z_0 \cup Z_1 \cup Z_2 \cup \bar{Y}_{1,2})$, *where* α, β_1, β_2 *are defined in (2.3).*

Theorem 2.1 is proved in Section 3.

Remark 2.1. Formulas (2.14), (2.15) of Theorem 2.1 remain valid with Γ_E in place of $\mathcal{M}_E$, where Γ_E is defined by (1.7). In addition, taking into account (2.9), (2.16) these formulas can be considered as explicit formulas for finding $\hat{v}$ on $\mathbb{R}^d$ from $S = \{|f|^2, |f_1|^2, |f_2|^2\}$ on Γ_Λ and background w_1, w_2 for any

$$\Lambda = \{E_j \in \mathbb{R}_+ : \quad j \in \mathbb{N}, \quad E_j \to \infty \ \text{ as } \ j \to \infty\}, \tag{2.18}$$

where Γ_Λ is defined in (1.11).

Corollary 2.1. *Let all assumptions of Theorem 2.1 on* v *and* w_1, w_2 *be fulfilled. Let* Λ *be defined as in (2.18). Then* $S = \{|f|^2, |f_1|^2, |f_2|^2\}$ *on* Γ_Λ *and background* w_1, w_2 *uniquely determine* v *in* $L^\infty(\mathbb{R}^d)$ *via formulas (2.14), (2.15) and the inverse Fourier transform.*

In addition to results of Theorem 2.1, Remark 2.1 and Corollary 2.1 on the explicit reconstruction from phaseless scattering data at high energies, we have also the following global uniqueness result for the case of finite energies:

Theorem 2.2. *Let* v *satisfy (1.2),* w_1, w_2 *satisfy (1.18), (2.16),* $d \geq 2$, *and* v, w_1, w_2 *be real-valued. Let*

$$\Lambda = \{E_j \in \mathbb{R}_+ : \quad j \in \mathbb{N}, \quad E_{j_1} \neq E_{j_2} \ \text{ for } \ j_1 \neq j_2, \ E_j \to E_* \ \text{ as } \ j \to \infty\}, \quad E_* > 0. \tag{2.19}$$

Then $S = \{|f|^2, |f_1|^2, |f_2|^2\}$ *on* Γ_Λ *and background* w_1, w_2 *uniquely determine* v *in* $L^\infty(\mathbb{R}^d)$.

Theorem 2.2 is proved in Section 4.

2.3. Results for the case $n = 1$.

Proposition 2.1. *If complex-valued* v *satisfies (1.2), complex-valued* w_1 *satisfies (1.18a),* $d \geq 2$, *then formulas (2.14), (2.15) hold for* $j = 0, 1$. *If, in addition,* $w_1 \neq 0$ *in* $L^\infty(\mathbb{R}^d)$, *then*

$$\begin{aligned}
&\cos(\alpha - \beta_1) = (2|\hat{v}||\hat{w}_1|)^{-1}(|\hat{v}_1|^2 - |\hat{v}|^2 - |\hat{w}_1|^2),\\
&\alpha = \alpha(p), \ |\hat{v}| = |\hat{v}(p)|, \ \beta_1 = \beta_1(p), \ |\hat{w}_1| = |\hat{w}_1(p)|, \ p \in \mathbb{R}^d \backslash (Z_0 \cup Z_1),
\end{aligned} \tag{2.20}$$

where α, β_1 *are defined in (2.3).*

Proposition 2.1 is proved in Section 3.

Proposition 2.2. *(A) There are not more than two different complex-valued potentials* v *satisfying (1.2) with given* $S = \{|f|^2, |f_1|^2\}$ *on* Γ_Λ *and background complex-valued* w_1 *satisfying (1.18a),* $w_1 \neq 0$ *in* $L^\infty(\mathbb{R}^d)$, *where* Λ *is defined as in (2.18). (B) There are not more than two different real-valued potentials* v *satisfying (1.2) with given* $S = \{|f|^2, |f_1|^2\}$ *on* Γ_Λ *and background real-valued* w_1 *satisfying (1.18a),* $w_1 \neq 0$ *in* $L^\infty(\mathbb{R}^d)$, *where* Λ *is defined as in (2.19).*

Proposition 2.2 is proved in Section 5.

3. Proofs of Proposition 2.1 and Theorem 2.1

3.1. Preliminaries. Let

$$L^\infty_\sigma(\mathbb{R}^d) = \{u \in L^\infty(\mathbb{R}^d) : \ \ \|u\|_\sigma < +\infty\},$$
$$\|u\|_\sigma = ess \sup_{x\in\mathbb{R}^d} (1+|x|^2)^{\sigma/2}|u(x)|, \ \ \sigma \geq 0. \tag{3.1}$$

Note that

$$v, \ w_j, \ v_j \in L^\infty_\sigma(\mathbb{R}^d) \ \text{ for each } \ \sigma \geq 0, \tag{3.2}$$

where v, w_j, v_j, $j \in \{1,2\}$, are the potentials of Proposition 2.1 and Theorem 2.1.

We recall that

$$\| < x >^{-s} G^+(k) < x >^{-s} \|_{L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)} \leq a_0(d,s)|k|^{-1},$$
$$k \in \mathbb{R}^d, \ \ |k| \geq 1, \ \ \text{for} \ \ s > 1/2, \tag{3.3}$$

where $G^+(k)$ denotes the integral operator with the Schwartz kernel $G^+(x-y,k)$ of (1.4), $< x >$ denotes the multiplication operator by the function $(1+|x|^2)^{1/2}$; see [E], [J] and references therein.

We will use the following detailed version of formula (1.12):

$$|f(k,l) - \hat{v}(k-l)| \leq 2(2\pi)^{-d}a_0(d,\sigma/2)(c_1(d,\sigma)\|v\|_\sigma)^2E^{-1/2},$$
$$(k,l) \in \mathcal{M}_E, \ \ E^{1/2} \geq \rho_1(d,\sigma,\|v\|_\sigma), \ \sigma > d, \tag{3.4}$$

where $a_0(d,s)$ is the constant of (3.3),

$$c_1(d,\sigma) = \Big(\int\limits_{\mathbb{R}^d} \frac{dx}{(1+|x|^2)^{\sigma/2}}\Big)^{1/2}, \tag{3.5}$$

$$\rho_1(d,\sigma,R) = \max(2a_0(d,\sigma/2)R, 1); \tag{3.6}$$

see formula (2.11) of [N7].

3.2. Proof of formulas (2.14), (2.15). We have that

$$|\hat{v}(k-l)| \overset{(3.1),(3.5)}{\leq} (2\pi)^{-d}\|v\|_\sigma(c_1(d,\sigma))^2, \tag{3.7}$$

$$||f(k,l)|^2 - |\hat{v}(k-l)|^2| = ||f(k,l)| - |\hat{v}(k-l)||(|f(k,l)| + |\hat{v}(k-l)|) \leq$$
$$|f(k,l) - \hat{v}(k-l)|(2|\hat{v}(k-l)| + |f(k,l) - \hat{v}(k-l)|), \tag{3.8}$$

where $(k,l) \in \mathcal{M}_E$, $\sigma > d$. Due to (3.4), (3.7), (3.8), we have that

$$||f(k,l)|^2 - |\hat{v}(k-l)|^2| \leq 3(2\pi)^{-d}\|v\|_\sigma(c_1(d,\sigma))^2|f(k,l) - \hat{v}(k-l)| \leq$$
$$6(2\pi)^{-2d}a_0(d,\sigma/2)((c_1(d,\sigma))^4(\|v\|_\sigma)^3E^{-1/2}, \tag{3.9}$$

$(k,l) \in \mathcal{M}_E$, $E^{-1/2} \geq \rho_1(d,\sigma,\|v\|_\sigma)$, $\sigma > d$. Formulas (2.14), (2.15) follow from (3.9) for $v = v_j$, $f = f_j$ and from the possibility of choice of $k = k_E(p)$, $l = l_E(p)$ as in (1.7) for $d \geq 2$. In addition,

$$c(D) = 6(2\pi)^{-2d} a_0(d,\sigma/2)(c_1(d,\sigma))^4(c_2(d,\sigma))^3, \tag{3.10}$$

$$\rho(D,N) = \rho_1(d,\sigma,c_2(D,\sigma)N), \tag{3.11}$$

for some fixed $\sigma > d$, where

$$c_2(D,\sigma) = \sup_{x\in D}(1+|x|^2)^{\sigma/2}. \tag{3.12}$$

3.3. Proof of formula (2.20). We have that

$$\begin{aligned} &|\hat{v}_1|^2 \overset{(1.17)}{=} |\hat{v}+\hat{w}_1|^2 \overset{(2.3)}{=} ||\hat{v}|e^{i\alpha} + |\hat{w}_1|e^{i\beta_1}| = \\ &(|\hat{v}|\cos\alpha + |\hat{w}_1|\cos\beta_1)^2 + (|\hat{v}|\sin\alpha + |\hat{w}_1|\sin\beta_1)^2 = \\ &|\hat{v}|^2 + |\hat{w}_1|^2 + 2|\hat{v}||\hat{w}_1|(\cos\alpha\cos\beta_1 + \sin\alpha\sin\beta_1) \quad \text{on} \quad \mathbb{R}^d. \end{aligned} \tag{3.13}$$

Formula (2.20) follows from (3.13).

3.4. Proof of formula (2.17). Using (3.13) and analogous formula for $\hat{v}_2 = \hat{v} + \hat{w}_2$, we obtain the system

$$\begin{pmatrix} \cos\beta_1 & \sin\beta_1 \\ \cos\beta_2 & \sin\beta_2 \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \begin{pmatrix} (2|\hat{v}||\hat{w}_1|)^{-1}(|\hat{v}_1|^2 - |\hat{v}|^2 - |\hat{w}_1|^2) \\ (2|\hat{v}||\hat{w}_2|)^{-1}(|\hat{v}_2|^2 - |\hat{v}|^2 - |\hat{w}_2|^2) \end{pmatrix}, \tag{3.14}$$

on $\mathbb{R}^d \backslash (Z_0 \cup Z_1 \cup Z_2)$.

Formula (2.17) follows from (3.14).

3.5. Final remark. Proposition 2.1 and Theorem 2.1 follow from formulas (2.14), (2.15), (2.20), (2.17) proved in Subsections 3.2, 3.3, 3.4.

4. Proof of Theorem 2.2

Let

$$\Delta_{E_0,E} = \{(k,l) \in \Gamma_E : \quad k - l \in \mathcal{B}_{2\sqrt{E_0}}\}, \quad 0 < E_0 \leq E, \tag{4.1}$$

where Γ_E, $\mathcal{B}_r$ are defined by (1.7), (1.8).

Theorem 2.2 follows from:

(1) the formulas of Theorem 2.1 with $\Delta_{E_*,E}$ in place of $\mathcal{M}_E$ and $|p| < 2\sqrt{E_*}$,

(2) the fact that $\hat{v}$ on $\mathcal{B}_{2\sqrt{E_*}} \backslash (Z_1 \cup Z_2 \cup \bar{Y}_{1,2})$ uniquely determines $\hat{v}$ on $\mathbb{R}^d$ (since $\hat{v}$ is real-analytic on $\mathbb{R}^d$), and

(3) the results of Lemma 4.1 for v and for $v = v + w_j$, $j = 1, 2$.

Lemma 4.1. *Let v satisfy (1.2) and be real-valued. Then:*

(a) $|f(k_E(p), l_E(p))|^2$ *is real-analytic in* $E \in]p^2/4, +\infty[$ *for fixed* $p \in \mathbb{R}^d$, *where* $k_E(p)$, $l_E(p)$ *are defined in (1.7);*

(b) $|f|^2$ on Γ_Λ uniquely determines $|f|^2$ on $\Delta_{E_*,E}$ for each $E \geq E_*$, where Γ_Λ, Λ are defined in (1.11), (2.19).

Statement (b) of Lemma 4.1 follows from statement (a) of Lemma 4.1 and the property that the accumulation point $E_* \in]p^2/4, +\infty[$ if $p \in \mathcal{B}_{2\sqrt{E_*}}$.

In turn, statement (a) of Lemma 4.1 follows from the presentation

$$|f|^2 = f\bar{f} \tag{4.2}$$

and from Lemma 4.2.

Lemma 4.2. *Let v satisfy (1.2) and be real-valued. Then $f(k_E(p), l_E(p))$ admits holomorphic extension in E to an open $\mathcal{N}$ in $\mathbb{C}$, where $]p^2/4, +\infty[\subset \mathcal{N}$, at fixed $p \in \mathbb{R}^d$.*

Lemma 4.2 follows from:

(1) the integral equation (1.4) for ψ^+ on D and the presentation (1.5) for f, where $k = k_E(p)$, $l = l_E(p)$;

(2) the property that

$$G^+(x,k) = G_0^+(|x|,|k|), \quad x \in \mathbb{R}^d, \quad k \in \mathbb{R}^d \backslash \{0\}, \tag{4.3}$$

where G^+ is the function of (1.4) and G_0^+ depends also on d;

(3) the properties that: $|k_E(p)| = E^{1/2}$ for $E \in]p^2/4, +\infty[$, $p \in \mathbb{R}^d$; $E^{1/2}$ is holomorphic in $E \in \mathbb{C}\backslash]-\infty, 0]$; $(E - p^2/4)^{1/2}$ is holomorphic in $E \in \mathbb{C}\backslash]-\infty, p^2/4]$; $p \in \mathbb{R}^d$; $G_0^+(r,\kappa)$ is holomorphic in $\kappa \in \mathbb{C}$ for odd $d \geq 3$ and in $\kappa \in \mathbb{C}\backslash]-\infty, 0]$ for even $d \geq 2$, where $r > 0$;

(4) the result that (1.4) with $k = k_E(p)$ is a Fredholm integral equation of the second kind for $\psi^+(\cdot,k) \in L^2(D)$ with holomorphic dependence on the parameter $E \in \mathbb{C}\backslash]-\infty, p^2/4]$ at fixed $p \in \mathbb{R}^d$;

(5) the result that (1.4) is uniquely solvable for $\psi^+(\cdot,k) \in L^2(D)$ for each $k \in \mathbb{R}^d \backslash \{0\}$ under our assumptions on v.

In connection with basic properties of function G^+ and basic properties of the Lippmann-Schwinger integral equation (1.4) we refer also to [BS], [F2], [Me].

5. Proof of Proposition 2.2

Proof of part (A). Due to formulas (2.14), (2.15) with Γ_E in place of $\mathcal{M}_E$, we have that $S = \{|f|^2, |f_1|^2\}$ on Γ_Λ uniquely determine $|\hat{v}|$, $|\hat{v}_1|$ on $\mathbb{R}^d$. If $|\hat{v}| \equiv 0$, then $v = 0$ in $L^\infty(\mathbb{R}^d)$. Therefore, it remains to consider the case when $|\hat{v}| \not\equiv 0$.

Due to (2.8), (2.9), $j = 1$, and continuity of $\hat{v}$, $\hat{w}_1$, we can choose $p' \in \mathbb{R}^d$, $r' > 0$ such that

$$|\hat{v}(p)| \neq 0, \; |\hat{w}_1(p)| \neq 0 \;\; \text{for} \;\; p \in \mathcal{B}_{p',r'}, \tag{5.1}$$

$$\mathcal{B}_{p',r'} = \{p \in \mathbb{R}^d : \;\; |p - p'| < r'\}. \tag{5.2}$$

Therefore, formula (2.20) for $\cos(\alpha - \beta_1)$ holds for each $p \in \mathcal{B}_{p',r'}$.

If $\cos(\alpha - \beta_1) \equiv 1$ on $\mathcal{B}_{p',r'}$, then $\alpha \equiv \beta_1 \; (mod\, 2\pi)$ on $\mathcal{B}_{p',r'}$. If $\cos(\alpha - \beta_1) \equiv -1$ on $\mathcal{B}_{p',r'}$, then $\alpha \equiv \beta_1 + \pi \; (mod\, 2\pi)$ on $\mathcal{B}_{p',r'}$. And in both cases $\hat{v} = |\hat{v}|e^{i\alpha}$ is uniquely determined on $\mathcal{B}_{p',r'}$ by $|\hat{v}|$, $|\hat{v}_1|$, $\hat{w}_1 = |\hat{w}_1|e^{i\beta_1}$ on $\mathcal{B}_{p',r'}$.

Due to continuity of $e^{i\alpha}$, $e^{i\beta_1}$ on $\mathcal{B}_{p',r'}$ we can choose $p'' \in \mathcal{B}_{p',r'}$ and $r'' \in]0, r'[$ such that

$$-1 < c_{min} \le \cos(\alpha - \beta_1) \le c_{max} < 1 \quad \text{on} \quad \mathcal{B}_{p'',r''} \tag{5.3}$$

for some fixed c_{min}, c_{max}. Therefore, due to formula (2.20) for $\cos(\alpha-\beta_1)$ and continuity of α, β_1 $(mod\, 2\pi)$ on $\mathcal{B}_{p'',r''}$ we have that either

$$\alpha = \beta_1 + arc\cos((2|\hat{v}||\hat{w}_1|)^{-1}(|\hat{v}_1|^2 - |\hat{v}|^2 - |\hat{w}_1|^2)) \tag{5.4a}$$

or

$$\alpha = \beta_1 - arc\cos((2|\hat{v}||\hat{w}_1|)^{-1}(|\hat{v}_1|^2 - |\hat{v}|^2 - |\hat{w}_1|^2)) \tag{5.4b}$$

$(mod\, 2\pi)$ on $\mathcal{B}_{p'',r''}$, where $arc\cos$ takes values in $[0,\pi]$. Therefore, there are not more than two different $\hat{v} = |\hat{v}|e^{i\alpha}$ on $\mathcal{B}_{p'',r''}$ with given $|\hat{v}|$, $|\hat{v}_1|$, $\hat{w}_1 = |\hat{w}_1|e^{i\beta_1}$ on $\mathcal{B}_{p'',r''}$. In turn, $\hat{v}$ on $\mathcal{B}_{p'',r''}$ uniquely determines $\hat{v}$ on $\mathbb{R}^d$ due to real analyticity of $\hat{v}$.

This completes the proof of part (A) of Proposition 2.2.

Proof of part (B). Due to statement (b) of Lemma 4.1 (for v and for $v = v + w_1$), $S = \{|f|^2, |f_1|^2\}$ on Γ_Λ uniquely determine S on $\Delta_{E_*,E}$. Due to formulas (2.14), (2.15) with $\Delta_{E_*,E}$ in place of $\mathcal{M}_E$ and $|p| < 2\sqrt{E_*}$, we have that S on $\Delta_{E_*,E}$ uniquely determine $|\hat{v}|$, $|\hat{v}_1|$ on $\mathcal{B}_{2\sqrt{E_*}}$.

Then in a completely similar way with the proof of part (A) of Proposition 2.2 we obtain that there are not more than two different $\hat{v}$ on $\mathcal{B}_{2\sqrt{E_*}}$ with given $|\hat{v}|$, $|\hat{v}_1|$, $\hat{w}_1$ on $\mathcal{B}_{2\sqrt{E_*}}$.

Finally, $\hat{v}$ on $\mathcal{B}_{2\sqrt{E_*}}$ uniquely determines $\hat{v}$ on $\mathbb{R}^d$ due to real analyticity of $\hat{v}$.

This completes the proof of part (B).

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